Extensions of Sets of Markov Operators Under Epistemic Irrelevance

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ISIPTA 2019
Ghent, 6 July 2019
Markov operators

Definition
Let $\mathcal{H}$ and $\mathcal{K}$ be sets of gambles. A Markov operator is a linear operator $T : \mathcal{H} \rightarrow \mathcal{K}$, such that:

1. if $f \leq g$ then $Tf \leq Tg$;
2. $T 1_{\mathcal{X}} = 1_{\mathcal{X}}$.

A set of Markov operators $\mathcal{T}$ can specify a probabilistic model by requiring:

1. if $f$ is a desirable gamble, then $Tf$ is also desirable for every $T \in \mathcal{T}$;
2. if $f$ is undesirable, then $T \in \mathcal{T}$ exists such that $Tf$ is undesirable.

A set of (almost) desirable gambles $\mathcal{D}$ satisfying the above requirements is said to be generated by $\mathcal{T}$.

In general, multiple such sets exist.
Examples

Sets of Markov operators are can be combined to achieve certain properties of probabilistic models.

Some motivating examples:

1. **Conditional expectation** $E_p(f|B)$ of a desirable gambles $f$ is usually also considered desirable. (If $p \in \mathcal{M}$ and $B = \{\emptyset, \mathcal{X}\}$, this model is equivalent to the model given by the credal set $\mathcal{M}$.)

2. **Symmetry** with respect to elements of $\mathcal{Y}$ for the gambles $f: \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$

   can be modelled by requiring that the set of desirable gambles is generated by the set $\mathcal{P}(\mathcal{Y})$ of all operators $T_\sigma$ of the form

   \[
   [P_\sigma f](x, y) = f(x, \sigma(y)),
   \]

   where $\sigma$ is a permutation on $\mathcal{Y}$. 
We consider specific question of extending a probabilistic model on $\mathcal{X}$ to a larger space, such as $\mathcal{X} \times \mathcal{Y}$ (or $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$).

<table>
<thead>
<tr>
<th>space</th>
<th>set</th>
<th>gambles</th>
<th>desirable gambles</th>
<th>operators</th>
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</thead>
<tbody>
<tr>
<td>extended space</td>
<td>$\mathcal{X}$</td>
<td>$\mathcal{G}(\mathcal{X})$</td>
<td>$\mathcal{D}(\mathcal{X})$</td>
<td>$\mathcal{T}$</td>
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<td></td>
<td>$\mathcal{X} \times \mathcal{Y}$</td>
<td>$\mathcal{G}(\mathcal{X} \times \mathcal{Y})$</td>
<td>$\mathcal{D}(\mathcal{X} \times \mathcal{Y})$</td>
<td>$\mathcal{T}$</td>
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How can we construct an extension $\mathcal{\tilde{T}}$ of $\mathcal{T}$ that generates $\mathcal{D}(\mathcal{X} \times \mathcal{Y})$?

The extension $\mathcal{\tilde{T}}$ depends on the type of extension $\mathcal{D}(\mathcal{X} \times \mathcal{Y})$ of $\mathcal{D}(\mathcal{X})$. 
Example – strong products

Take for example the **strong product** of credal sets $\mathcal{M}$ and $\mathcal{N}$ on separate spaces $\mathcal{X}$ and $\mathcal{Y}$:

<table>
<thead>
<tr>
<th>space</th>
<th>set</th>
<th>credal sets</th>
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<tbody>
<tr>
<td>extended space</td>
<td>$\mathcal{X} \times \mathcal{Y}$</td>
<td>$\mathcal{M} \times \mathcal{N}$</td>
<td>$\mathcal{T} \otimes \mathcal{S}$</td>
</tr>
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where

$$\mathcal{T} \otimes \mathcal{S} = \{ T \otimes S : T \in \mathcal{T}, S \in \mathcal{S} \}$$

and $\otimes$ denotes the **tensor product** of operators.

**Note:**

The structure of the model, i.e. independent product, must be pre-specified – not every $\mathcal{T} \otimes \mathcal{S}$-generated credal set is an independent product of its marginals.
Epistemic irrelevance

A set of desirable gambles \( D \subset G(\mathcal{X} \times \mathcal{Y}) \) satisfies epistemic irrelevance \( \mathcal{Y} \to \mathcal{X} \) if for every \( \mathcal{X} \)-measurable gamble \( f \) the following conditions are equivalent:

- \( f \in D \);
- \( l_y f \in D \) for every \( y \in \mathcal{Y} \).

Remark

A gamble \( f \) is \( \mathcal{X} \)-measurable if \( f(x, y) = f(x, y') \) for every \( y, y' \in \mathcal{Y} \).

Epistemic irrelevance combines two properties:

- symmetry w.r.t. \( \mathcal{Y} \);
- the sum \( f = \sum_{y \in \mathcal{Y}} l_y f \) is only desirable if all \( l_y f \) are desirable.
Example – symmetry alone does not imply irrelevance

Let $\mathcal{X} = \mathcal{Y} = \{0, 1\}$.
Consider the following gambles:

$$
\begin{array}{c|cc}
Y = 0 & Y = 1 \\
X = 0 & -1 & -1 \\
X = 1 & 1 & 1 \\
\end{array}
$$

which is $\mathcal{X}$-measurable;

$$
\begin{array}{c|cc}
Y = 0 & Y = 1 \\
X = 0 & 0 & -1 \\
X = 1 & 0 & 1 \\
\end{array}
$$

$\mathcal{X}$-measurable;
Take two probability mass functions:

\[
\begin{array}{c|cc}
  & Y = 0 & Y = 1 \\
 X = 0 & 3/16 & 3/16 \\
 X = 1 & 2/16 & 8/16 \\
\end{array}
\]

\[
\begin{array}{c|cc}
  & Y = 0 & Y = 1 \\
 X = 0 & 3/16 & 3/16 \\
 X = 1 & 8/16 & 2/16 \\
\end{array}
\]

Let \( \mathcal{M} = \{p_1, p_2\} \) and then

\[
P_\mathcal{M}(f) = 4/16, \quad P_\mathcal{M}(I_{Y=0}f) = -1/16, \quad P_\mathcal{M}(I_{Y=1}f) = -1/16.
\]

So we have that

- \( I_{Y=0}f \) and \( I_{Y=1}f \) are undesirable,
- while \( f = I_{Y=0}f + I_{Y=1}f \) is desirable.

Thus, even if there is clear symmetry with respect to \( \mathcal{Y} \), we do not have epistemic irrelevance.
Additive independent extension

In addition to symmetry we need a new property to guarantee epistemic irrelevance:

**Definition (Additive independent extension)**

A set $\mathcal{D}$ is an additive independent extension of $\{\mathcal{D}_i \subset \mathcal{G}_i\}_{i \in I}$ if

- $f \in \mathcal{D}$;
- $f = \sum_{i \in I} f_i$, $f_i \in \mathcal{G}_i$

imply that $\exists i \in I : f_i \in \mathcal{D}_i$.

A sum of undesirable gambles cannot be desirable.
Let $\mathcal{T}$ be a set of Markov operators on $\mathcal{G}(\mathcal{X})$.

We construct the following set of Markov operators:

$$\tilde{\mathcal{T}} = \left\{ \tilde{T} : \tilde{T} f = \sum_{y \in \mathcal{Y}} I_y T_y f(\cdot, y), T_y \in \mathcal{T} \forall y \in \mathcal{Y} \right\}.$$  

Restricted to $\mathcal{G}(\mathcal{X}|y)$ that is isomorphic to $\mathcal{G}(\mathcal{X})$, $\tilde{\mathcal{T}}$ acts as $\mathcal{T}$.

Hence, the generated sets of desirable gambles $\mathcal{D}(\mathcal{X}|y)$ are all generated by $\mathcal{T}$, yet are not necessarily equal, as $\mathcal{T}$ can have multiple generated models.

So, we additionally need to require symmetry, which we can do by adding the set of permutation operators.
Main result

Let $\mathcal{D}$ be a set of desirable gambles that is an additive independent extension of $\{\mathcal{D} \cap \mathcal{G}(\mathcal{X}|y) : y \in \mathcal{Y}\}$. Then the following are equivalent:

1. $\mathcal{D}$ satisfies epistemic irrelevance $\mathcal{Y} \rightarrow \mathcal{X}$;
2. $\mathcal{D}$ is generated by $\tilde{T} \cup \mathcal{P}(\mathcal{Y})$;
3. two sets of desirable gambles:
   - $\mathcal{D}(\mathcal{X})$ : generated by $\mathcal{T}$ and
   - $\mathcal{D}(\mathcal{Y})$ : generated by $\mathcal{P}(\mathcal{Y})$ (only required to be symmetric w.r.t. $\mathcal{Y}$)
exist so that every $f \in \mathcal{D}$ can be written in the form:

$$f = f_\mathcal{Y} + \sum_{y \in \mathcal{Y}} I_y f_y,$$

where $f_\mathcal{Y} \in \mathcal{D}(\mathcal{Y})$ and $f_y \in \mathcal{D}(\mathcal{X})$ for every $y \in \mathcal{Y}$. 
Further work

- The advantage of operators approach is that the probabilistic model (even imprecise) does not need to be fully specified – we can only have a set of (local) conditional models, which we can easily extend to larger (global) probability spaces.

- Moreover, additional requirements, such as symmetry, can easily be given in terms of additional sets of operators.

- This can be useful in models where compatible probabilistic models need to be constructed based on conditional models only.

- Such examples include stochastic processes, credal networks, probability trees and others.

- In particular, it would be interesting to impose requirements such as time homogeneity or Markov property in terms of Markov operators.