On Minimum Elementary-triplet Bases for Independence Relations

Janneke H. Bolt and Linda C. van der Gaag

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Probabilistic independence relations

A set of triplets $\langle A, B \mid C \rangle$ with $A, B, C \subset V$, where a triplet $\langle A, B \mid C \rangle$ captures that

$$Pr(A, B \mid C) = Pr(A \mid C) \cdot Pr(B \mid C)$$

for all possible value combinations of $A, B, C$.

Not any subset of all possible triplets $V^{(3)}$ is a probabilistic independence relation. For example, since $\langle A, B \mid C \rangle$ implies $\langle B, A \mid C \rangle$, each probabilistic independence relation will either include none or both of these triplets.
Semi-graphoid axioms

$G1$: if $\langle A, B \mid C \rangle$ then $\langle B, A \mid C \rangle$

$G2$: if $\langle A, BD \mid C \rangle$ then $\langle A, B \mid C \rangle$

$G3$: if $\langle A, BD \mid C \rangle$ then $\langle A, B \mid CD \rangle$

$G4$: if $\langle A, B \mid CD \rangle$ and $\langle A, D \mid C \rangle$ then $\langle A, BD \mid C \rangle$

A semi-graphoid independence relation is a subset of triplets $\overline{J} \subseteq V^{(3)}$ that satisfies the above properties for all sets $A, B, C, D \subseteq V$.

A semi-graphoid independence relation $\overline{J}$ can be inferred from a starting set of triplets $J$ by repeatedly applying the semi-graphoid axioms.
Elementary triplets

Triples of the form $\langle A, B \mid C \rangle$.

A semi-graphoid relationship is fully captured by its elementary triplets.

Semi-graphoid axioms for elementary triplets:

$E1$: if $\langle A, B \mid C \rangle$ then $\langle B, A \mid C \rangle$

$E2$: if $\langle A, B \mid CD \rangle$ and $\langle A, D \mid C \rangle$ then $\langle A, B \mid C \rangle$ and $\langle A, D \mid CB \rangle$
Bases for semi-graphoid independence relations

- Dominant triplets. Any triplet of the independence relation can be derived from one triplet in the basis through axioms G1-G3.
- Elementary triplets.

For example:

<table>
<thead>
<tr>
<th>Dominant basis</th>
<th>Elementary basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle 1, 2 \mid \emptyset \rangle$</td>
<td>$\langle 1, 2 \mid \emptyset \rangle$</td>
</tr>
<tr>
<td>$\langle 1, 2 \mid 3 \rangle$</td>
<td>$\langle 1, 2 \mid 3 \rangle$</td>
</tr>
<tr>
<td>$\langle 1, 3 \mid \emptyset \rangle$</td>
<td>$\langle 1, 3 \mid \emptyset \rangle$</td>
</tr>
<tr>
<td>$\langle 1, 3 \mid 2 \rangle$</td>
<td>$\langle 1, 3 \mid 2 \rangle$</td>
</tr>
<tr>
<td>$\langle 1, {2, 3} \mid \emptyset \rangle$</td>
<td>$\langle 1, {2, 3} \mid \emptyset \rangle$</td>
</tr>
</tbody>
</table>
Minimum elementary triplet bases

Redundant information in an elementary triplet basis.

\( E2: \) if \( \langle A, B \mid CD \rangle \) and \( \langle A, D \mid C \rangle \) then \( \langle A, B \mid C \rangle \) and \( \langle A, D \mid CB \rangle \)

For example, the elementary triplet basis
\{\langle 1, 2 \mid \emptyset \rangle, \langle 1, 2 \mid 3 \rangle, \langle 1, 3 \mid \emptyset \rangle, \langle 1, 3 \mid 2 \rangle \} \) can be reduced to
\{\langle 1, 2 \mid \emptyset \rangle, \langle 1, 3 \mid 2 \rangle \}

Minimally needed:

- All \( A, B \)-combinations present in the independence relation.
- All cardinalities of \( C \) present in the independence relation.

Nb.

- A minimum elementary triplet basis is not unique.
- One by one removal of triplets does not necessarily yield a minimum basis.
Minimum bases for singleton starting sets

The semi-graphoid closure of the triplet

\[ \langle \{ A_1, \ldots, A_n \}, \{ B_1, \ldots, B_m \} \mid C \rangle \]

is also represented by the elementary triplets

\[ \{ \langle A_i, B_j \mid A \setminus \{ A_i, \ldots A_n \} \cup B \setminus \{ B_j, \ldots B_m \} \cup C \rangle \mid i = 1, \ldots, n, j = 1, \ldots, m \} \]

For example, the semi-graphoid closure of \( \langle \{1, 2\}, \{3, 4\} \mid \emptyset \rangle \) is represented by the elementary triplets

\[ \{ \langle 1, 3 \mid \emptyset \rangle, \langle 1, 4 \mid 3 \rangle, \langle 2, 3 \mid 1 \rangle, \langle 2, 4 \mid \{1, 3\} \rangle \} \]

This implies that the semi-graphoid closure of \( \langle A, B \mid C \rangle \) can be represented by \( |A| \cdot |B| \) elementary triplets.
A few questions

- How efficient are minimum elementary triplet bases compared to dominant triplet bases?
- How to compute a minimum basis efficiently?
- Is one by one removal of triplets a good heuristic?